Optimal Backward Error and the Dahlquist Test Problem

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Abstract

The Dahlquist test problem is the simple linear first order ODE $\dot{y} = \lambda y$, $y(t_n) = y_n$, to be solved on $t_n \leq t \leq t_n + h$. Taking $\lambda \in \mathbb{C}$ gives rise to the classical theory of stiffness (when $\mathrm{Re}\lambda \ll 0$). Several mathematical theories emerged from study of this equation: A-stability, L-stability, and the so-called theory of "order stars". These theories connected approximation theory over $\mathbb{C}$ with the theory of stability of numerical methods.

Recently, we have made some progress understanding numerical methods for initial-value problems (IVP) $\dot{y} = f(y)$, $y(0) = y_0$ from the point of view of "optimal backward error". That is, a numerical method, say $y_{n+1} = y_n + h\Phi$, produces a *skeleton* $\{(t_n, y_n)\}_{n=0}^{N}$ of a solution, which is usually interpolated by a piecewise polynomial to give a continuously differentiable solution. By interpolating instead with the solution of either

$$\dot{z} = f(z) + \Delta(t) \qquad t_n \leq t_{n+1}$$

or

$$\dot{z} = f(z)(1 + \delta(t)) \qquad t_n \leq t_{n+1}$$

we ask for the "smallest" Δ (or δ) that still has both $z(t_n) = y_n$ and $z(t_{n+1}) = y_{n+1}$. This computational *a posteriori* analysis can be applied to any problem and any method. For the Dahlquist test problem, the results are surprising. This talk uses the calculus of variations and optimal control theory. But only just.