Applied Math 505 Project: Stress Propagation in a Granular Column

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May 20, 2006

Abstract

This paper uses a biharmonic partial differential model derived from continuum mechanics to study stress propagation in granular materials [1]. Nonlinearity is introduced by expressing the density as a nonlinear function of the stress [2]. Appropriate boundary conditions are chosen based on knowledge obtained from performing numerical hard (non-deformable) sphere collision simulations. The hard spheres are modelled as an isotropic material and together with a perturbation technique, the solution to the nonlinear partial differential equation problem entails solving two linear problems. First, separation of variables [3] is used to solve a product of Laplace equations for the zero order perturbative epsilon term and second, Fourier transforms and a specialized method [3] are used to solve an inhomogeneous biharmonic equation for subsequent order perturbative epsilon terms. Finally, a finite difference scheme for solving the biharmonic equation [4] is introduced as an alternative numerical approach that can be used to compare with the analytical results.
1 Introduction

The following introductory material is from [1]. The stress tensor \( \sigma \) is related to the gradients of the vector field \( u \) by

\[
\sigma_{ij} = \lambda_{ijkl} u_{kl},
\]

where \( \sigma_{ij} \) is a component of the stress tensor, \( u_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \), and \( \lambda_{ijkl} \) are the elastic constants. The general equilibrium equations are:

\[
\partial_i \sigma_{ij} = \rho(\sigma_{jj}) g_j,
\]

where \( \rho(\sigma_{ij}) \) is a nonlinear stress dependent density and \( g_j \) is the acceleration due to gravity. A supplementary condition of compatibility is:

\[
\partial_2^2 z u_{xx} + \partial_2^2 x u_{zz} - 2 \partial_x \partial_z u_{xz} = 0,
\]

We can relate the stress tensor \( \Sigma = (\sigma_{xx}, \sigma_{zz}, \sigma_{xz})^T \) to the strain tensor \( U = (u_{xx}, u_{zz}, u_{xz})^T \) giving a matrix equation:

\[
\Sigma = \Lambda U,
\]

where

\[
\Lambda = \begin{pmatrix}
\lambda_{xxx} & \lambda_{xzz} & 2\lambda_{xxxz} \\
\lambda_{zzxx} & \lambda_{zzzz} & 2\lambda_{zzxz} \\
\lambda_{xxxx} & \lambda_{zzxx} & 2\lambda_{zzxx}
\end{pmatrix}.
\]

To express the compatibility relation in terms of the stress tensor, we express \( U \) in terms of \( \Sigma \):

\[
U = B \Sigma,
\]

where \( B = (B_{ij}) = \Lambda^{-1} \). Then Eq. (3) is rewritten as:

\[
B_{1j} \partial_2^2 z \Sigma_j + B_{2j} \partial_2^2 x \Sigma_j - 2B_{3j} \partial_x \partial_z \Sigma_j = 0.
\]

If we choose \( x \) and \( z \) to be along the principal axes, we have:

\[
\Lambda = \begin{pmatrix}
a & c & 0 \\
c' & b & 0 \\
0 & 0 & d
\end{pmatrix} = \frac{1}{1 - \nu_x \nu_z} \begin{pmatrix}
E_x & \nu_z E_x & 0 \\
\nu_x E_x & E_z & 0 \\
0 & 0 & (1 - \nu_x \nu_z) G
\end{pmatrix},
\]

where \( E_{x,z} \) and \( G \) are the Young and shear moduli, respectively, and \( \nu_{x,z} \) the Poisson ratios. Note in classical theory \( c = c' \) resulting in the extra relation \( \frac{E_z}{E_x} = \frac{\nu_x}{\nu_z} \).
The compatibility condition (7) then reads:

\[ b \partial_z^2 \sigma_{xx} - c \partial_z^2 \sigma_{zz} - c' \partial_z^2 \sigma_{xx} + a \partial_z^2 \sigma_{zz} - 2 \frac{\det \Lambda}{d^2} \partial_x \partial_z \sigma_{xx} = 0. \] (9)

Combining this equation with the two equilibrium conditions of Eq. (2),

\[ \partial_z \sigma_{zz} + \partial_x \sigma_{xz} = -\rho(\sigma_{zz}) g, \] (10)

\[ \partial_z \sigma_{xz} + \partial_x \sigma_{xx} = 0, \] (11)

we obtain for any one of the stress tensor components,

\[ (\partial^4_z + t \partial^4_x + 2r \partial^2_x \partial^2_z) \sigma_{ij} + (2s \partial^2_x + \partial^2_z) g \partial_z (\rho(\sigma_{zz})) = 0, \] (12)

where

\[ t = \frac{a}{b} = \frac{E_x}{E_z}, \] (13)

\[ r = \frac{ab - cc' - \frac{1}{2}d(c + c')}{bd} = \frac{1}{2} E_x \left( \frac{2}{G} - \frac{\nu_z}{E_z} - \frac{\nu_x}{E_x} \right), \] (14)

\[ s = \frac{ab - cc' - \frac{1}{2}d c'}{bd} = \frac{E_x}{G} - \frac{1}{2} \nu_x. \] (15)

A derivation of Eq. (12) is given in Appendix A.

For isotropic hard spheres, we have \( t = r = s = 1 \). Thus we have:

\[ (\partial^4_z + \partial^4_x + 2 \partial^2_x \partial^2_z) \sigma_{ij} + \epsilon (2 \partial^2_x + \partial^2_z) g \partial_z (\rho(\sigma_{zz})) = 0, \] (16)

where we have perturbed the nonlinear \( \rho(P = \sigma_{zz}) \) term which can take on a form such as in [2]:

\[ \rho(P = \sigma_{zz}) = \frac{\rho_0}{1 - \left( \frac{\rho_\infty - \rho_0}{\rho_\infty} \right) \left\{ c \left[ 1 - \exp \left( \frac{-P}{P_l} \right) \right] + (1 - c) \exp \left( \frac{-P}{P_m} \right) \right\}. \] (17)

Eq. (17) is derived in Appendix B.

Note that it makes sense to perturb the density, for when we simulated our granular system, the density was almost constant in the dense, glassy region. See Figs. 1 and 2.

The solution then becomes:

\[ \sigma_{ij} = \sigma_{ij0} + \sigma_{ij1} \epsilon + \sigma_{ij2} \epsilon^2 + \ldots, \] (18)

To keep things simple, we will solve a two-dimensional system where \( x \) is in the left to right horizontal direction and \( z \) is in the vertical
direction, and assume isotropy in the front to back \( y \) direction. We will only solve for \( \sigma_{zz} \) and call it \( u = \sigma_{zz} \), since solving for the other two stresses can be solved using Eqs. (10) and (11), once \( \sigma_{zz} \) is known. The boundary conditions are obtained by studying results from a granular column simulation (see Fig. 1) which this problem addresses. Details of the simulation and a description of how the stresses were calculated in the simulation are given in Appendix C.

![Figure 1: granular column](image)

The boundary conditions for the dense, glassy portion of the granular simulation are taken as:

\[
0 < x < w, \ 0 < z < h, \quad \text{(19)}
\]

\[
\partial_z u(x, z)|_{z=h'} = 0, \quad \text{see Fig. 3} \quad \text{(20)}
\]

\[
u(x, 0) = \delta(x - x'), \quad \text{reflecting sieve at bottom} \quad \text{(21)}
\]

\[
u(0, z) = 0, \quad \text{reflecting left wall, so no vertical stress} \quad \text{(22)}
\]

\[
u(w, z) = 0, \quad \text{reflecting right wall, so no vertical stress}. \quad \text{(23)}
\]

where \( h' \) is in the dense, glassy region, and \( 0 < x' < w \).

Note that an additional boundary condition that was also observed in the dense portion of the simulation, exists for \( \sigma_{xz} \):

\[
\partial_x \sigma_{xz}(x, z)|_{z=h'} = \text{constant}, \quad \text{see Fig. 4} \quad \text{(24)}
\]
Figure 2: Density profile

Figure 3: $\partial_z \sigma_{zz} \approx 0$ in glassy region
Figure 4: $\partial_x \sigma_{zz} \approx \text{constant in glassy region}$

2 Solving terms of zero order epsilon

If we solve for terms of zero order epsilon we have the linear partial differential equation:

$$(\partial^4_x + \partial^4_z + 2\partial^2_x \partial^2_z)u_0 = 0, \quad (25)$$

This equation can be factored yielding:

$$(\partial^2_x + \partial^2_z)(\partial^2_x + \partial^2_z)u_0 = 0, \quad (26)$$

Thus we are left with solving the Laplace equation for $u_0$.

We will now use separation of variables as presented in [3]. We first let $u_{0k} = M_k(x)N_k(z)$. Then we find that:

$$N_k(z) = \hat{a}_k \exp\left[\frac{\pi k}{w}z\right] + \hat{b}_k \exp\left[-\frac{\pi k}{w}z\right] \quad (27)$$

The functions $N_k(z)$ can be expressed as linear combinations of hyperbolic sine functions and we write $u_{0k}(x,z)$ as:

$$u_{0k}(x,z) = \left\{a_k \sinh\left[\frac{\pi k}{w}z\right] + b_k \sinh\left[\frac{\pi k}{w}(z - h)\right]\right\} \sqrt{\frac{2}{w}} \sin\left(\frac{\pi k}{w}x\right),$$

$$u_0(x,z) = \sum_{k=1}^{\infty} u_{0k}(x,z). \quad (28)$$
The boundary condition at \( z = h' \) implies:
\[
\partial_z u(x, z)\big|_{z=h'} = 0 = \sum_{k=1}^{\infty} \frac{\pi k}{w} \left( a_k \cosh \left( \frac{\pi k h'}{w} \right) + b_k \cosh \left( \frac{\pi k (h' - h)}{w} \right) \right) \sqrt{\frac{2}{w}} \sin \left( \frac{\pi k}{w} x \right).
\]

Therefore
\[
a_k = - \frac{b_k \cosh \left( \frac{\pi k (h' - h)}{w} \right)}{\cosh \left( \frac{\pi k h'}{w} \right)}.
\]

Similarly the boundary condition at \( z = 0 \) implies:
\[
u(x, 0) = \delta(x - x') = \sum_{k=1}^{\infty} b_k \sinh \left( - \frac{\pi k h}{w} \right) \sqrt{\frac{2}{w}} \sin \left( \frac{\pi k}{w} x \right).
\]

Therefore,
\[
b_k = \frac{\left( \delta(x - x'), \sqrt{\frac{2}{w}} \sin \left( \frac{\pi k}{w} x' \right) \right)}{\sinh \left( - \frac{\pi k h}{w} \right)}
\]
\[
= \frac{\sqrt{\frac{2}{w}} \sin \left( \frac{\pi k}{w} x' \right)}{\sinh \left( - \frac{\pi k h}{w} \right)}.
\]

### 3 Solving terms of first order epsilon

Now that we have a solution for \( u_0(x, z) = \sigma_{zz0}(x, z) \), we can use it to solve for \( u_1(x, z) = \sigma_{zz1}(x, z) \) in the perturbation expansion Eq. (18).

If we solve for terms of first order epsilon we have the linear partial differential equation:
\[
(\partial_z^4 + \partial_x^4 + 2\partial_z^2 \partial_x^2)u_1 + (2\partial_x^2 + \partial_z^2)g \partial_z (\rho(u_0)) = 0,
\]

where we can use the chain rule:
\[
g \partial_z (\rho(u_0)) = g \partial_u (\rho(u))|_{u=u_0} \partial_z u_0.
\]

Effectively, \((2\partial_x^2 + \partial_z^2)g \partial_z (\rho(u_0))\) will then be a function of \( x \) and \( z \) since we know from solving the terms of zero order epsilon \( u_0 \) as a function of \( x \) and \( z \). We will set:
\[
(2\partial_x^2 + \partial_z^2)g \partial_z (\rho(u_0)) = f(x, z),
\]

7
which results in the equation:

\[ (\partial_{x}^{4} + \partial_{z}^{4} + 2\partial_{x}^{2}\partial_{z}^{2})u_{1} = -f(x, z). \]  

(37)

We will assume that \( u_{1}(x, z) \) satisfies the additional condition:

\[ u_{1}(x, z) \rightarrow 0 \text{ as } |x|, |z| \rightarrow \infty, \]  

(38)

since for the perturbed terms the stresses are assumed to be so small that we may assume an unbounded region.

We will now use Fourier transforms as presented in [3]. We will first assume that \( f(x, z) \) has a Fourier transform. Then if we apply the two dimensional Fourier transform to (37) we easily obtain:

\[ U(\vec{\lambda}) = -\frac{F(\vec{\lambda})}{(\lambda_{1}^{2} + \lambda_{2}^{2})^{2}}, \]  

(39)

where \( \vec{\lambda} = [\lambda_{1}, \lambda_{2}] \) and \( U(\vec{\lambda}) \) and \( F(\vec{\lambda}) \) are the Fourier transforms of \( u_{1}(x, z) \) and \( f(x, z) \), respectively. We have assumed in obtaining (39) that not only \( u_{1} \) but also its partial derivatives vanish at infinity.

Inverting the transform gives the solution:

\[ u_{1}(x, z) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\lambda_{1}x + \lambda_{2}z)} \frac{F(\vec{\lambda})}{(\lambda_{1}^{2} + \lambda_{2}^{2})^{2}} d\lambda_{1} d\lambda_{2}. \]  

(40)

Since \( F(\vec{\lambda}) \) has the form:

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\lambda_{1}\xi + \lambda_{2}\eta)} f(\xi, \eta) d\xi d\eta, \]  

(41)

then

\[ u_{1}(x, z) = -\frac{1}{4\pi^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{-i [\lambda_{1}(x - \xi) + \lambda_{2}(z - \eta)]\} \times \frac{f(\xi, \eta)}{(\lambda_{1}^{2} + \lambda_{2}^{2})^{2}} d\lambda_{1} d\lambda_{2} d\xi d\eta, \]  

(42)

on interchanging the order of integration.

Introducing polar coordinates in the two inner integrals in (42), that is

\[
\begin{align*}
\lambda_{1} &= \rho \cos \phi, \quad \lambda_{2} = \rho \sin \phi, \\
x - \xi &= r \cos \theta, \quad z - \eta = r \sin \theta,
\end{align*}
\]  

(43) 

(44)
we readily obtain
\[ \lambda_1 (x - \xi) + \lambda_2 (z - \eta) = \rho r \cos (\phi - \theta). \] (45)

Then transforming to polar coordinates gives
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ -i \left[ \lambda_1 (x - \xi) + \lambda_2 (z - \eta) \right] \right\} \frac{d\lambda_1}{(\lambda_1^2 + \lambda_2^2)^2} \frac{d\lambda_2}{\rho^4} = \int_0^{2\pi} \int_0^\infty e^{-i\rho r \cos(\phi - \theta)} \rho d\phi d\rho, \quad (46)
\]
where we have used
\[ (\cos^2 \phi + \sin^2 \phi)^2 = 1. \quad (47) \]

Using a known representation of the Bessel function of order zero \( J_0(z) \),
\[ J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \cos(\phi - \theta)} d\phi, \quad (48) \]
where \( \theta \) is constant. In fact in the granular column it is known experimentally and from simulations that the stresses propagate along \( \theta = \pm \frac{\pi}{4} \). See Fig. 5 [5].
Figure 5: Experimental [5] stress propagation in granular column along $\theta = \pm \frac{\pi}{4} = 45^\circ$
Since \( J_0(z) = J_0(-z) \), we have:
\[
\int_0^\infty \int_0^{2\pi} e^{-i\rho r \cos(\phi - \theta)} \rho d\phi d\rho = 2\pi \int_0^\infty \frac{J_0(\rho r)}{\rho^2} d\rho = 2\pi r^2 \int_0^\infty J_0(\rho r) \frac{d(\rho r)}{(\rho r)^3}.
\]
(49)

Actually our infinity is the width of the granular column \( w \), so we are left with solving
\[
2\pi r^2 \int_0^w J_0(\rho r) \frac{d(\rho r)}{(\rho r)^3}.
\]
(51)

Maple 10 evaluated this as an indefinite integral (letting \( y = \rho r \)) as:
\[
2\pi r^2(-4yJ_0(y)LommelS1(-4, 1, y) + yJ_1(y)LommelS1(-3, 0, y)),
\]
(52)

where \( LommelS1 \) is defined as:
\[
LommelS1(a, b, z) = -z^{(a+1)} \frac{\text{hypergeom}([1], [\frac{3}{2} + \frac{a}{2}, \frac{3}{2} - \frac{b}{2}, \frac{a}{2}, -\frac{z^2}{4}])}{(-a + b - 1)(a + b + 1)},
\]

\[\text{And}(-a + b - 1 \neq 0, a + b + 1 \neq 0, (3/2 - 1/2 * b + 1/2 * a) :: \text{Not(nonposint)}),\]
\[ (3/2 + 1/2 * b + 1/2 * a) :: \text{Not(nonposint)}), (53)\]

The \text{hypergeom} in the above definition is the hypergeometric function which is defined as:
\[
\text{hypergeom} ([A], [B, C], x) = 1 + \frac{A \cdot B}{1 \cdot C} x + \frac{A(A + 1)B(B + 1)}{1 \cdot 2 \cdot C(C + 1)} x^2 + \frac{A(A + 1)(A + 2)B(B + 1)(B + 2)}{1 \cdot 2 \cdot 3 \cdot C(C + 1)(C + 2)} x^3 + ... \]
(55)

Then we can solve for \( u_1(x, z) \) as
\[
u_1(x, z) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty r^2(-4prJ_0(\rho r) \times LommelS1(-4, 1, \rho r) + \rho rJ_1(\rho r)LommelS1(-3, 0, \rho r))|_{\rho = \rho(z, \xi)} f(\xi, \eta) d\xi d\eta,
\]

with the limits of \( \rho \) evaluated between a small lower limit \( \epsilon \) and the upper limit \( w \) of the granular column \( w \), and where \( r = \sqrt{(x - \xi)^2 + (z - \eta)^2} \).

The fact that we have Bessel functions in our solution is not surprising, because in a competing model that uses the telegrapher’s equation to model stresses in granular systems, its solution has modified Bessel functions [6].
4 Alternative method for solving terms of first order epsilon

An alternative method of solving Eq. (37) as presented in [3] exists for the special case where \( f(x, z) \) is a point source. In this method, we integrate over a region \( R \) with boundary \( \partial R \) that contains a point \( P_0 = (x_0, z_0) \) in its interior and obtain

\[
\int \int_R \nabla^2 \nabla^2 u dxdz = \int \int_R \nabla \cdot (\nabla \nabla^2 u) dxdz
\]
\[
= \int_{\partial R} \frac{\partial}{\partial n} (\nabla^2 u) ds
\]
\[
= - \int \int_R f dxdz
\]
\[
= -1. \tag{56}
\]

Here we have used the divergence theorem and chosen \( f(x, z) \) as a point source or, equivalently, a Dirac delta function with singularity at \( P_0 \) so that the integral of \( f \) equals unity.

Introducing polar coordinates with the pole at \( P_0 \) and taking the limit \( \partial R \to P_0 \), we have

\[
\lim_{\partial R \to P_0} \int_{\partial R} \frac{\partial}{\partial n} (\nabla^2 u) ds = \lim_{r \to 0} \int_0^{2\pi} \frac{\partial}{\partial r} (\nabla^2 u) |_{x = x_0 + r \cos \theta} \ r d\theta \]
\[
= \lim_{r \to 0} \left[ 2\pi r \frac{\partial}{\partial r} (\nabla^2 u) \right]
\]
\[
= -1, \tag{57}
\]

where \( R \) is a disk of radius \( r \) centered at \( P_0 \).

We next assume that \( u \) is a function of \( r \) only resulting in

\[
\nabla^2 u(r) = \frac{1}{r} \frac{\partial}{\partial R} \left( r \frac{\partial u}{\partial r} \right). \tag{58}
\]

We thus obtain the approximate equation

\[
2\pi r \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial R} \left( r \frac{\partial u}{\partial r} \right) \right] \approx -1. \tag{59}
\]

Integrating out this equation and keeping only the most singular terms yields the solution

\[
u = -\frac{1}{8\pi} r^2 \log r, \tag{60}\]
where \( r^2 = (x - x_0)^2 + (z - z_0)^2 \).

It is to be noted that (60) is a solution of the biharmonic equation everywhere except at \( r = 0 \).

5 Solving terms of second and higher order epsilon

Now that we have solved for \( u_1(x, z) \), we can solve for \( u_2(x, z) \) by solving for the second order terms of epsilon using:

\[
(\partial_x^4 + \partial_z^4 + 2\partial_x^2\partial_z^2)u_2 = -f_1(x, z),
\]

(61)

where

\[
f_1(x, z) = (2\partial_x^2 + \partial_z^2)g\partial_z(\rho(u_1))
\]

(62)

and the process repeats itself for higher order epsilon terms, that is we have a recurrence relation:

\[
(\partial_x^4 + \partial_z^4 + 2\partial_x^2\partial_z^2)u_n = -f_{n-1}(x, z),
\]

(63)

where

\[
f_{n-1}(x, z) = (2\partial_x^2 + \partial_z^2)g\partial_z(\rho(u_{n-1}))
\]

(64)

6 Numerical approach

So far we have pursued a purely analytical treatment. One alternative is to discretize the rectangular domain of the granular column and use a finite difference scheme to solve for the higher order epsilon terms. A 13 point finite difference formula exists for the biharmonic equation [4]:

\[
\nabla^2\nabla^2 u = 20u(i, j) - 8 \times (u(i + 1, j) + u(i - 1, j) + u(i, j + 1) + u(i, j - 1))
+ 2(u(i + 1, j + 1) + u(i - 1, j + 1) + u(i - 1, j - 1) + u(i + 1, j - 1))
+ (u(i + 2, j) + u(i - 2, j) + u(i, j + 2) + u(i, j - 2)) + O(h^4)
\]

(65)

This numerical approach can be used to solve for the higher order epsilon terms, which we can use to compare with the analytical results.
7 Conclusion

This paper has provided a perturbative analytical treatment for studying nonlinear stress propagation in a granular column. It has also presented a useful numerical finite difference scheme. It is the intention of this author to continue investigating these ideas as part of his Ph.D. thesis. One extension of this model is to solve the anisotropic case for a general \( t, r \) and \( s \). Analytically, here one could use the Fourier transform scheme outlined in [1] applying the boundary conditions and perturbation method outlined in this paper. Numerically, one would use a finite difference scheme such as:

\[
\left(\partial_z^4 + t\partial_x^4 + 2r\partial_x^2\partial_z^2\right)u = (6t + 8r + 6)u \\
-(4t + 4r)\left[u(i + 1, j) + u(i - 1, j)\right] - (4 + 4r)\left[u(i, j + 1) + u(i, j - 1)\right] \\
+2r\left[u(i + 1, j + 1) + u(i - 1, j - 1) + u(i - 1, j + 1) + u(i + 1, j - 1)\right] \\
+t\left[u(i + 2, j) + u(i - 2, j)\right] + u(i, j + 2) + u(i, j - 2) + O\left(h^4\right)
\] (66)

and with the nonlinear density term use the perturbation scheme.

Also when polydisperse (different sized) spheres are used the dense region remains randomly close packed, but with monodisperse (same sized) spheres the dense region starts out being randomly close packed but over time the system crystallizes [8]. A recent experimental study has shown that spatial ordering is a key factor in the force response [9]. We would like to test this model in these two regimes to see how the stresses propagate in randomly close packed versus crystallized regions.

A Appendix A

Here we derive the biharmonic equation for the stresses. We have

\[
\partial_z\sigma_{zz} + \partial_x\sigma_{xz} = -\rho(\sigma_{zz})g \\
\partial_z\sigma_{zx} + \partial_x\sigma_{xx} = 0.
\] (67) (68)

If we apply \( \partial_z \) to (67), we have

\[
\partial_z^2\sigma_{zz} + \partial_z\partial_x\sigma_{xz} = -\frac{\partial}{\partial\sigma_{zz}}[\rho(\sigma_{zz})] \frac{\partial\sigma_{zz}}{\partial z}g
\] (69)

Similarly, if we apply \( \partial_x \) to (68), we have

\[
\partial_x\partial_z\sigma_{xz} + \partial_x^2\sigma_{xx} = 0.
\] (70)
As discussed in the introduction section, we also have the compatibility equation (9):

\[ b \partial_z^2 \sigma_{xx} - c \partial_z^2 \sigma_{zz} - c' \partial_z^2 \sigma_{xx} + a \partial_z^2 \sigma_{zz} - 2 \frac{(ab - cc')}{d} \partial_x \partial_z \sigma_{xx} = 0. \] (71)

If we apply \( \frac{1}{b} \partial_z^2 \) to (71), we have

\[ \partial_x^2 \[ - \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} \] \]

Then, if we apply \( \partial_z^2 \) to (69), we have

\[ \partial_z^2 \partial_x^2 \sigma_{zz} + \partial_z \partial_x^3 \sigma_{xx} = \partial_x \left[ - \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} \right] \] \]

Similarly, if we apply \( \partial_z^2 \) to (70), we have

\[ \partial_z^2 \partial_x^2 \sigma_{xx} + \partial_z \partial_x^3 \sigma_{xx} = 0. \] (73)

Subtracting (74) from (73), we have

\[ \partial_x^2 \partial_z^2 \sigma_{zz} - \partial_x^4 \sigma_{xx} = \partial_x \left[ - \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} \right] \] (75)

Also, (73) implies that

\[ \partial_x^2 \partial_z \sigma_{xx} = - \partial_x^2 \partial_z^2 \sigma_{zz} - \partial_x \left[ - \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} \right] \] (76)

Together with (72), we then have

\[ \partial_x^2 \partial_z^2 \sigma_{xx} - \frac{c}{b} \partial_x^2 \partial_z^2 \sigma_{zz} - \frac{c'}{b} \partial_x^4 \sigma_{xx} + \frac{a}{b} \partial_x^4 \sigma_{zz} + \frac{2(ab - cc')}{bd} \left[ \partial_x^2 \partial_z^2 \sigma_{xx} - \partial_x^2 \left[ - \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} \right] \right] = 0 \] (77)

Now if we apply \( \partial_z^2 \) to (69), we have

\[ \partial_z^4 \sigma_{zz} + \partial_z^3 \partial_x \sigma_{xx} = \partial_x \left[ - \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} \right] \] (78)

Similarly, if we apply \( \partial_z^2 \) to (70), we have

\[ \partial_x \partial_z^3 \sigma_{xx} + \partial_z^2 \partial_z^2 \sigma_{xx} = 0. \] (79)
Subtracting (79) from (78), we have
\[ \partial^4_z \sigma_{zz} - \partial^2_x \partial^2_z \sigma_{xx} = -\partial^2_z \left[ \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} g \right] \] (80)

Therefore
\[ \partial^2_x \partial^2_z \sigma_{xx} = \partial^4_z \sigma_{zz} + \partial^2_z \left[ \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} g \right] \] (81)

Also from (75) we have
\[ \partial^4_x \sigma_{xx} = \partial^2_x \partial^2_z \sigma_{xx} + \partial^2_x \left[ \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} g \right] \] (82)

Substituting (81) and (82) into (77) we have
\[ \partial^4_z \sigma_{zz} + \partial^2_x \partial^2_z \sigma_{zz} + \partial^2_x \left[ \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} g \right] - \frac{c}{b} \partial^2_x \partial^2_z \sigma_{zz} - \frac{c'}{b} \left[ \partial^2_x \partial^2_z \sigma_{zz} + \partial^2_x \left[ \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} g \right] \right] + \frac{a}{b} \partial^4_x \sigma_{zz} + 2 \frac{ab - cc'}{bd} \left[ \partial^2_x \partial^2_z \sigma_{zz} + \partial^2_x \left[ \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z} g \right] \right] = 0 \] (83)

Letting
\[ w = g \frac{\partial}{\partial \sigma_{zz}} \left[ \rho(\sigma_{zz}) \right] \frac{\partial \sigma_{zz}}{\partial z}, \] (84)
\[ u = \sigma_{zz}, \] (85)

we finally have
\[ \partial^4_z u + \frac{a}{b} \partial^4_z u + 2 \left[ \frac{ab - cc' - \frac{1}{2}d(c + c')}{bd} \right] \partial^2_x \partial^2_z u + 2 \left[ \frac{ab - cc' - \frac{1}{2}d(c + c')}{bd} \right] \partial^2_x w + \partial^2_x w = 0 \] (86)
B Appendix B

Here we derive the nonlinear density-stress relation as quoted from [2]. "The probability \( f \) of a subsystem in a [granular system such as a] powder overcoming a volume barrier, resulting in a void being filled, is given by

\[
\begin{align*}
    f &= (\text{constant}) \exp \left( -\frac{P_a}{P} \right),
\end{align*}
\]

where \( P_a \) is a single pressure barrier. As the pressure \( P = \sigma_{zz} \) increases, little effect is observed for values of \( P < P_a \). As \( P \) exceeds \( P_a \), an abrupt increase in the probability of void filling occurs. Physically, this happens because a deformation process, such as crushing, chipping, or fracture, has occurred. ...

Also to be considered is another process that is not pressure activated; this is the simple rearrangement obtained during the initial stage of compaction. The number of voids filled is proportional, for this process, to the factor \( f_l \):

\[
\begin{align*}
    f_l &= (\text{constant}) \left[ 1 - \exp \left( -\frac{P}{P_l} \right) \right].
\end{align*}
\]

\( P_l \) is the saturation value of the pressure at which further increases in pressure have no effect during that process. ...

Taking into consideration the fact that these two types of processes occur simultaneously—i.e., that the number of voids being filled will be given by a weighed sum of the right-hand sides of Eqs. (87) and (88), viz., \( c \left[ 1 - \exp(-P/P_l) \right] + (1 - c)\exp(-P_a/P) \)—we obtain the relation:

\[
\begin{align*}
    \rho(\rho = \sigma_{zz}) &= \frac{\rho_0}{1 - \left( \frac{\rho_\infty - \rho_0}{\rho_\infty} \right) \left\{ c \left[ 1 - \exp \left( -\frac{P}{P_l} \right) \right] + (1 - c) \exp \left( -\frac{P_a}{P} \right) \right\}}.
\end{align*}
\]

Here, \( \rho_\infty \) is the theoretical density achievable only at infinite value of the pressure \( P \), and \( \rho_0 \) is the initial value of the compact density at zero pressure. In obtaining Eq. (89), we have assumed (for simplicity) that a single pressure barrier \( P_a \) exists, that the rearrangement process also is characterized by a single value of \( P_l \), and that the rearrangement process and the activated process occur in the relative ratio \( c : (1 - c) \). Thus, \( c \) is the amount of compaction because of rearrangement."
C Appendix C

Here is a description of how the stresses were calculated in the computer simulation.

The simulation models hard sphere particles colliding with a normal restitution coefficient \( \mu \) as they fall down a rectangular chute under the influence of gravity as shown in the introduction section in Fig. 1. At the bottom of the chute a sieve is modeled by having the particles reflecting from the bottom of the chute with a probability \( p \) and transmitting through the bottom of the chute with a probability \( 1 - p \). Unless otherwise specified, in our simulations \( p \) was 10%. Particles that transmit through the bottom of the chute reappear at the top of the chute once again to fall down through it. Particles reflect off the left and right walls of the chute with a partial loss, 10%, in their vertical velocity. This is a simple effective way to model rough walls. Depending on the type of simulation that is desired, there are periodic or reflecting front and back walls.

In the simulation, the velocities after collision \( \dot{r}_1' \) and \( \dot{r}_2' \), expressed in terms of the velocities before collision, \( \dot{r}_1 \) and \( \dot{r}_2 \), are

\[
\begin{pmatrix}
\dot{r}_1' \\
\dot{r}_2'
\end{pmatrix} = \begin{pmatrix}
\dot{r}_1 \\
\dot{r}_2
\end{pmatrix} + \frac{1}{2} (1 + \mu) \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
\dot{r}_1 \cdot q \\
\dot{r}_2 \cdot q
\end{pmatrix} q,
\]

where \( q = (r_2 - r_1) / |r_2 - r_1| \), and \( \mu \) is the coefficient of restitution.

\( \mu \) is a velocity-dependent restitution coefficient described by the phenomenological fit to experiments,

\[
\mu (v_n) = \begin{cases}
1 - (1 - \mu_0) \left( \frac{v_n}{v_0} \right)^{0.7}, & v_n < v_0 \\
\mu_0, & v_n > v_0
\end{cases}
\]

Here \( v_n \) is the component of relative velocity along the line joining the grain centers, \( v_0 = \sqrt{2g\alpha} \) and \( \mu_0 \) varying between 0 and 1 is the asymptotic coefficient at large velocities (typically 0.9 here). \( \alpha \) is the radius of the particle and \( g \) is the acceleration due to gravity.

A transfer of momentum from the colliding particles results from changes in velocities before and after a collision. Surprisingly, it is not so much the magnitude of this momentum but the collision direction \( q \) joining the centers of colliding particles that determines the normal and shear stresses in the granular system.

Momentum conservation requires that

\[
\partial_t (\rho v_i) + \partial_k (\rho v_i v_k) = \partial_k \sigma_{ik} + \rho g_i
\]

(92)
where $\sigma_{ik}$ is the stress tensor, and all $v_k$ refer to first moments of the velocity distribution. For our steady state situation in the glassy state the left hand side is zero.

The stresses are numerically calculated by incorporating the momentum transfer with the collision directions and are averaged over time as shown in the formula below:

$$
\sigma_{ij} = \frac{1}{t} \sum_{\text{collisions}} \frac{1}{2} (1 + \mu) (\dot{r}_1 - \dot{r}_2) \cdot \hat{q} \left( \begin{array}{ccc}
(q \cdot \hat{x})^2 & (q \cdot \hat{x})(q \cdot \hat{y}) & (q \cdot \hat{x})(q \cdot \hat{z}) \\
(q \cdot \hat{y})(q \cdot \hat{x}) & (q \cdot \hat{y})^2 & (q \cdot \hat{y})(q \cdot \hat{z}) \\
(q \cdot \hat{z})(q \cdot \hat{x}) & (q \cdot \hat{z})(q \cdot \hat{y}) & (q \cdot \hat{z})^2
\end{array} \right),
$$

$$
\approx -\frac{1}{2} f_c (1 + \mu) (\dot{r}_1 - \dot{r}_2) \cdot \hat{q} \left( \begin{array}{ccc}
\langle(q \cdot \hat{x})^2\rangle & \langle(q \cdot \hat{x})(q \cdot \hat{y})\rangle & \langle(q \cdot \hat{x})(q \cdot \hat{z})\rangle \\
\langle(q \cdot \hat{y})(q \cdot \hat{x})\rangle & \langle(q \cdot \hat{y})^2\rangle & \langle(q \cdot \hat{y})(q \cdot \hat{z})\rangle \\
\langle(q \cdot \hat{z})(q \cdot \hat{x})\rangle & \langle(q \cdot \hat{z})(q \cdot \hat{y})\rangle & \langle(q \cdot \hat{z})^2\rangle
\end{array} \right).
$$

In the above equation, the sum is over collisions in a (long) time interval $t$, $f_c$ is the collision frequency and $\mu$ is a velocity-dependent coefficient of restitution. In writing this we assumed that we could separate the factor $(1 + \mu) (\dot{r}_1 - \dot{r}_2) \cdot \hat{q}$ from the factors in the matrix when computing averages. The factors in front of the matrix appear to be nearly a constant, hence the structure of the stress tensor comes almost entirely from the collision directions.

References


