Solve $y'' + y = \delta\left(t - \frac{\pi}{2}\right) + \delta\left(t - \frac{3\pi}{2}\right)$, $y(0) = 0$, $y'(0) = 0$

First we solve for the homogeneous or complementary solution: $y'' + y = 0$, $y(0) = 0$, $y'(0) = 0$

Letting $y = e^{rt}$, $r^2 + 1 = 0$, $r = \pm i$

$y_c = a\cos t + b\sin t$

Applying the initial conditions:

$y(0) = 0 \to a = 0$ and $y'(0) = 0 \to b = 0$.

Thus $y_c = 0$.

The Green function $G(t, t_0) = G(t - t_0)$ is

$$G(t, t_0) = \begin{cases} A\cos t + B\sin t, & \text{for} \quad t < t_0 : \ t^- \\ C\cos t + D\sin t, & \text{for} \quad t_0 < t : \ t^+ \end{cases} \tag{1}$$

Applying the initial conditions to the Green function:

$y(0) = 0 \to G(0, t_0) = 0 \to A\cos 0 + B\sin 0 = 0 \to A = 0$,

$y'(0) = 0 \to G_t(t, t_0)|_{t=0} = 0 \to -A\sin 0 + B\cos 0 = 0 \to B = 0$.

Applying the jump condition to the Green function:

$G_t|_{t=t^+} - G_t|_{t=t^-} = \frac{1}{a_2} = \frac{1}{1} = 1$ (a_2 is the coefficient of y'')

Thus $-C\sin t_0 + D\cos t_0 - (-A\sin t_0 + B\cos t_0) = 1$

But $A = 0$ and $B = 0$. Thus $-C\sin t_0 + D\cos t_0 = 1$.

Applying continuity to the Green function $G(t^+, t_0) = G(t^-, t_0)$:

$C\cos t_0 + D\sin t_0 = A\cos t_0 + B\sin t_0$

But $A = 0$ and $B = 0$. Thus $C\cos t_0 + D\sin t_0 = 0$.

Solving for C and D using Cramer's rule, we have $C = -\sin t_0$ and $D = \cos t_0$.

Thus

$$G(t, t_0) = \begin{cases} 0, & \text{for} \quad t < t_0 : \ t^- \\ -\sin t_0 \cos t + \cos t_0 \sin t = \sin(t - t_0), & \text{for} \quad t_0 < t : \ t^+ \end{cases} \tag{2}$$

Thus

$$\begin{aligned} y(t) &= \int_0^1 G(t, t_0) f(t) dt \\ &= \int_0^{t_0} G(t, t_0) f(t) dt + \int_{t_0}^1 G(t, t_0) f(t) dt \\ &= \int_0^{t_0} 0 \cdot f(t) dt + \int_{t_0}^1 \sin(t - t_0) \left[\delta(t - \frac{\pi}{2}) + \delta(t - \frac{3\pi}{2})\right] dt \\ &= 0 + \sin\left(t - \frac{\pi}{2}\right) U\left(t - \frac{\pi}{2}\right) + \sin\left(t - \frac{3\pi}{2}\right) U\left(t - \frac{3\pi}{2}\right), \end{aligned} \tag{3}$$

where U is the unit step function.

But $\sin\left(t - \frac{\pi}{2}\right) = \sin t \cos\frac{\pi}{2} - \cos t \sin\frac{\pi}{2} = -\cos t$,

and $\sin\left(t - \frac{3\pi}{2}\right) = \sin t \cos\frac{3\pi}{2} - \cos t \sin\frac{3\pi}{2} = \cos t$.

Thus $y(t) = -\cos t \ U(t - \frac{\pi}{2}) + \cos t \ U(t - \frac{3\pi}{2})$.