Computer-Mediated Thinking

Robert M. Corless
ORCCA and the Department of Applied Mathematics
University of Western Ontario
London, CANADA
Rob.Corless@uwo.ca

July, 2004

Abstract

This paper discusses *computer-mediated thinking* and some of its possible implications for curriculum design in mathematics education. We begin with a discussion of today's context and of ideas related to computer-mediated thinking. We continue with examples of the use of computer-mediated thinking in modern applied mathematics. We then extract some suggestions for a curriculum in mathematics centred at the calculus level. We include specific suggestions for removing material from the current syllabus. We end with a discussion of the *unintentional power* of the calculus.

1 Context

Computer-mediated thinking is not necessarily the same as *computer-mediated teaching*. One may use technology to teach the same things that were taught before technology became available. Indeed, one may use technology to teach those things *better*, by promoting active learning or (for instance) by using technology to implement 'just-in-time-teaching' [14]. This is what some people call 'computer-mediated teaching'.

What I mean by 'computer-mediated thinking' is something different; it is what Andrea di Sessa calls 'material intelligence' [11] and what Peter Jones calls an 'intelligent partnership' [16]. We will see details of several mathematical examples of this later in the paper, but for now here are instances of computer-mediated thinking from several fields.

Computational Materials Physics One use of computational tools is to visualize atoms and their interactions and thereby make them part of our experience; this can be done in *no other way* [Martin Müser, private communication].

Applied Ethics: Peter Danielson uses *agents* programmed with simple rules, game theory, and simulations to investigate the emergence (or non-emergence) of ethical behaviour in groups. This leads to insights into human behaviour that are not easily accessible by introspection or by inspection of the historico-political record. See `http://www.ethics.ubc.ca/people/danielson/`

Computer-generated music: In his Nerenberg lecture "Chaotic music and fractal art: a glimpse into the neurophysiology of aesthetics", Leon Glass (McGill University) discussed computer programs that generated 'synthetic Mozart': genuinely emotionally meaningful music generated by a computer program that contained a database of Mozart's works and chaotic dynamics for randomness. The result was surprisingly pleasant, and very reminiscent of Mozart. This is qualitatively different from arranging and playing Bach on a synthesizer, as W. Carlos did (so beautifully). Unfortunately, I do not have the original reference and cannot cite the person who wrote this program and/or composed the music, mediated of course by computer.

Computer-aided proofs and discovery of new mathematics: The most famous computer-aided proof is that of the four-colour theorem, by Appel and Haken in 1977 [1, 2]. There have now been many mathematical theorems proved by computer. I find it more interesting that genuinely new mathematics has arisen from computational studies: for instance, solitons, complexity, chaotic dynamics, and new combinatorial identities and methods for solving recurrence relations and summing series. See in particular Doron Zeilberger's home page `http://www.math.rutgers.edu/~zeilberg/` and his opinions.

1.1 Counterexamples: the map is not the territory

> *"Sir, I decided to help you, and so I ran your circuit in SPICE. Your circuit doesn't work."*—a student to Mark W. Tilden (the father of BEAM Robotics). Mark's wordless response: he took down a robot using the circuit in question, turned it on and showed the student that *it was the SPICE simulation that was wrong*.

> *"Because of a wonky software package called S+, many EPA studies on smoking-related deaths have been called into question."* — an unfair newspaper article (the convergence test in S+ did indeed have a bad default value, but it is the *users* of the package who were at fault.)

We are all aware of the danger of using computation as a *substitute* for thought—students computing an example and considering that suffices as proof, for instance. The danger is real, though guessing and verification by sampling go a lot farther towards proof than a skeptic might think: consider the prevalence of Monte Carlo and other probabilistic algorithms, which work in a fashion that allows the user to bound the probability of failure. But sometimes we may be misled by computation; see for example [4].

> Some people think they want this, but they are wrong.
>
> —Kelly Roach

There is also computer-mediated *avoidance* of thought. By this I mean using a computer or wireless handheld device to cheat with (instant messaging, email, SMS, file sharing, essay/exam services, etc.); these force us to use electronic countermeasures. There are also the annoying facts of life of computers: malicious viruses, scumware and adware; gratuitous software and hardware "updates" and "upgrades"; and incompatibilities amongst the myriad constellations of equipment surrounding both us and our students.

This is just life.

1.2 Let us begin as we mean to go on

This paper is not about teaching styles; you will already know that there is a great deal of evidence now that teaching methods can be dramatically improved by using active learning [11, 14]. This paper is also not about progress in the cognitive psychology of learning, through which we may build theoretical models of what it is we want to do for students. See [12], for example, or Uri Leron's recent work on the dissonance between social (natural) reasoning and mathematical reasoning [18].

> *[Learning mathematics consists of]*
> *a gradual and painful reversal of perspective.* —Uri Leron

It is my belief that technology simply extends this process—mathematicians have themselves had to painfully reverse their own perspective when confronted with the task of implementing their theorems. This is where the concept of "Lies to Children" comes in (see *The Science of Discworld* by Terry Pratchett, Ian Stewart, and Jack Cohen). *A "lie to children" is a useful oversimplification that starts one on the path to better knowledge.* It is a truth unacknowledged that mathematics before computers was a lie to children.

This paper *is* about teaching mathematics "on shifting sand" (*cf.* [17]); the way people do and use mathematics is changing more rapidly now than in any previous era. Regardless of teaching style, regardless of how people learn, we have to recognize this, and deal with it.

This paper is also about my opinion of what topics should be taught in the near future, and how to accommodate the changes to come. I make no prescriptions as to how this material should be taught—I am more concerned with the content of the curriculum in the ubiquitous presence of technology.

2 Examples

2.1 Summation of series

Recent work by Zeilberger, Wilf, Gosper, Petkovšek, and others has given rise to the fully automated solution of several problems that formerly required significant ingenuity. For example, one may ask Maple to sum series that are beyond the reach of most first-year students: *e.g.*,

$$\sum_{n=1}^{N} \frac{n^2}{2^n} = -4\ (1/2)^{N+1}\ (N+1) - 6\ (1/2)^{N+1} - 2\ (1/2)^{N+1}\ (N+1)^2 + 6$$

and, perhaps more surprisingly,

$$S(x) = \sum_{n\geq 1} \frac{x^n}{n^2} = \mathrm{polylog}(2, x)$$

which introduces a function doubtless unknown to the first-year student (albeit very well described in the beautiful book [19]). Numerical evaluation of the sum in Maple at some random point, say $x = -3$, gives $S(-3) = -1.93937542\ldots$, while evaluation of the function gives $\mathrm{polylog}(2, -3.0) = -1.9393754207667\ldots$, in perfect agreement.

At this point there might be a little head-scratching on the part of the reader: $x = -3$ is outside the circle of convergence for that series, but Maple did not cavil at the sum. Indeed, even Maple's *numerical* summation technique, which uses Levin's u-transform to speed up convergence, also got the right answer, agreeing with the polylog function, which is the *Euler sum* of the series.

Not only has Maple introduced a function unknown to the student, it has introduced a new meaning for the summation symbol. These are both good things (albeit surprising). This will be taken up again later. For now, I leave you with two apparently antithetical quotes:

> The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. —N. H. Abel

> This series is divergent. Therefore, we may be able to do something with it. —O. Heaviside

2.2 Eigenvalues are answers, not questions

Eigenvalues and eigenvectors are deeply useful. The word *eigen* in German has several meanings, but the relevant ones for matrices are "innate", "own", and (in the old sense of 'characteristic') "peculiar". Eigenvalues and eigenvectors are innate characteristics of a matrix (say A): an eigenvalue-eigenvector pair is a scalar (say λ) and a vector (say x) such that, to the vector x, the matrix A acts like a scalar, in that $Ax = \lambda x$. More generally, matrix *pencils* (A, B)—a peculiar way of saying pairs (A, B)—have *generalized* eigenvalues and eigenvectors: $\alpha Ax = \beta Bx$ for some scalars α and β. If $\alpha \neq 0$ then we can put $\lambda = \beta/\alpha$, but if $\alpha = 0$ then we say that the eigenvalue is infinite.

Eigenvalues are taught to students by considering the *characteristic polynomial* $\det(A - \lambda I)$ or $\det(\alpha A - \beta B)$; the zeros of the polynomial are the eigenvalues, and once the eigenvalues are known it is a simple job to compute the eigenvectors.

There's no such thing as a simple job. —Tim Daly

Eigenvalues and eigenvectors have physical meaning. Possibly the most important is in the theory of vibration. A structural system (say a building in the wind, or an airplane wing, or an undersea cable) has *mass* and *stiffness*; if the vibration is small and the damping can be neglected, then we can often write the equations of motion of the structural system as $M\ddot{x} = -Kx$ using the *mass matrix* M and the *stiffness matrix* K. The eigenvalues of this pencil are the natural frequencies of vibration (actually the squares of the natural frequencies), and the eigenvectors depict the physical "mode shape" of vibration.

Eigenvalues and eigenvectors are so important in applied mathematics that many hundreds of person-years have gone into creating reliable, efficient software for their computation. Now we come to the rub: as an *afterthought*, the problem-solving environment Matlab included a `roots` function to compute polynomial roots. It worked by constructing a companion matrix whose eigenvalues were the roots and then using its eigenvalue routine! For example, one companion matrix for $p(x) = x^5 - 4x^4 - 3x^3 - 2x^2 - x - 9$ is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 9 & 1 & 2 & 3 & 4 \end{bmatrix}$$

as can be verified by computation[1] of $\det(xI - A)$.

This approach makes sense if you already have an eigenvalue solver, and was taken up by the HP series of calculators. The TI calculator did polynomial roots first,by a stable and efficient method known as Laguerre's method [David Stoutemyer, private communication], and added eigenvalues afterward, which seems more logical, and efficient in use of space and time.

But it turns out that in a technical sense *eigenvalues are easier to compute* than polynomial roots—they are less sensitive to changes in their data: in the jargon, polynomial roots are often *ill-conditioned* (an old phrase that meant rude, boorish, or ill-mannered, two hundred years ago), whereas eigenvalues are usually *well-conditioned*. Improvements in hardware and software mean that efficiency considerations hardly apply any more. Even in Maple (formerly notoriously slow) one may compute eigenvalues of an 800 by 800 matrix in 35 seconds on a battery-powered laptop. Therefore, nowadays, the sensitivity aspects are more important.

Here is where many mathematicians had to (slowly and painfully) reverse their perspective. It is eigenvalues that are more geometrically fundamental, more frequent in applications, and less sensitive to inevitable errors in the data. So, in a certain sense, eigenvalues are *answers*, not questions.

Let me give an example from my own work. I have recently invented and used a new formulation of the companion matrix, for polynomials expressed in

[1] Dare you do this by hand? I did! It's not so bad if you pick the correct place to expand about. If you succeed, you will strongly believe the result.

the Lagrange basis, to develop a new method for finding zeros of polynomials given just by samples of their values, without first constructing an interpolating polynomial [7, 10]. Suppose for example that we know a polynomial $p(x)$ is of degree 3 and that we are given its values at 4 points: at $x = [-2, -1, 1, 2]$ we have that $p = [-3, 2, -1, 1]$. Consider the matrix pencil

$$C_0 = \begin{bmatrix} -2 & 0 & 0 & 0 & -3 \\ 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 1/12 & -1/6 & 1/6 & -1/12 & 0 \end{bmatrix}$$

and

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

I claim $\det(x_k C_1 - C_0) = y_k$ for $0 \le k \le 3$, and that $\deg\det(xC_1 - C_0) = 3$. That is, by the interpolation theorem, this determinant is exactly the interpolating polynomial for the data. Therefore the finite eigenvalues of this pencil are the roots of the polynomial, which we have not constructed explicitly.

By Maple, the eigenvalues are

$$\lambda = -1.6070196422161,\ 0.41720813126321,\ 1.7898115109530\,,$$

and two at infinity (we started with a 5 by 5 matrix, and the polynomial is degree 3). The polynomial roots are correct to one unit in the last place, each.

Testing this method on roots of polynomials given by the values either $+1$ or -1 on the 12th roots of unity (there are 2^{11} such polynomials with different roots: therefore there are $12 \cdot 2^{11} = 24,576$ possibly different roots) gives the picture in Fig. 1. The symmetry we observe induces confidence that the answers are right; moreover, the method computed all these roots in seconds.

The point is that the method would not have made sense to even try, before eigenvalues were answers, i.e. before technology.

2.3 Special Functions

In the early days of numerical analysis it was predicted that special functions (such as the error function, the Gamma function, the Riemann zeta function) would go the way of the dodo, because direct numerical computation and solution of the relevant differential or difference equations would fill all the needs for these functions. It hasn't turned out that way, for a number of reasons, and

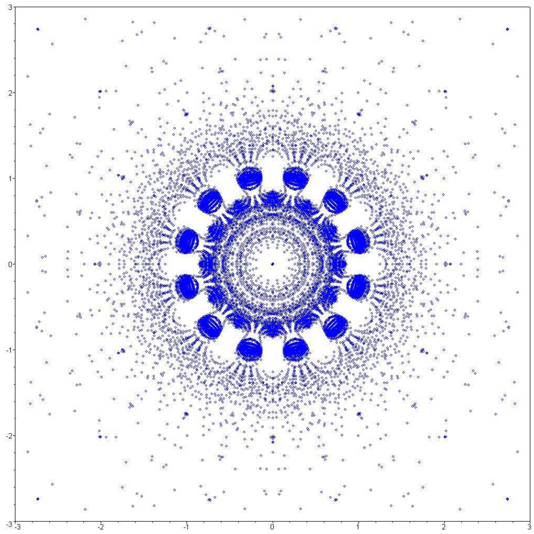

Figure 1: All roots of polynomials taking on values ± 1 at the 12th roots of unity.

most people who know and love the theory and practice of functions are very glad. In this case, it is the numerical analysts who have had to reverse their perspective.

> Civilisation advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in battle—they are strictly limited in number, they require fresh horses, and must only be made at decisive moments.
> —A. N. Whitehead, *Introduction to Mathematics*, Williams and Norgate, 1911

Named functions help scientists to think. This is probably their most important role. By carrying a name and having a notation they indicate that there is a body of known properties (usually internalized into the scientist's thinking habits). The exponential function, the sine function, and the logarithm are of course the simplest examples, and therefore the most useful; but it is surprising how much of science depends on the shorthand notation offered by functions.

A recent success of computer algebra is the introduction of a notation and a name for a function that had been in substantive use without a name; the resulting economy of thought has brought an explosion of applications and new thinking. I refer to the Lambert W function, surveyed in [8]. This is nothing more or less than the solution of the transcendental equation

$$ye^y = x$$

for y, giving (by definition) $y = W(x)$.

My favourite example of an application of this function is the beautiful paper [20], in which the Lambert W function is used to distinguish between two models of recovery of the human eye after exposure to brilliant light.

2.4 More on infinite series

In the paper [21], we find the authors advocating the use of conversion of infinite series to differential equations, and evaluation of the sum by numerical solution of the differential equation. The problem they seek to avoid goes by various names in the literature (for example, "the hump"). Work by J. C. P. Miller, W. Gautschi, and many others, analyzes the problem that the paper [21] is so concerned about: namely that even convergent infinite series can be difficult to evaluate.

The simplest example is one that J. M. Varah used (probably original to G. Forsythe) when he taught me numerical analysis in 1978: the exponential function. Consider evaluating $\exp(-30)$ by using Taylor's series:

$$e^{-30} = 1 - 30 + \frac{30^2}{2!} - \cdots + \frac{(-30)^k}{k!} + \cdots .$$

Of course, this series is convergent, and, mathematically, the error will be less than (for example) 10^{-23} if we omit terms with $k \geq 123$ (it's an alternating series). It is also numerically useless, because the largest terms, occurring for $k = 30$ and $k = 31$, have magnitude about $7 \cdot 10^{11}$; whereas the final answer must be $e^{-30} \doteq 9.357622969 \cdot 10^{-14}$. Therefore, we need about 35 decimal digits to accurately sum all these terms. Many calculators and computers use the IEEE-750 standard, with at most 80 bits or 18 decimals of precision. Summing this series naively using such arithmetic will produce garbage (in 14 digits in Maple I get an answer of -0.02225302, which doesn't even have the right sign, has no digits correct, and isn't even the right order of magnitude).

> "In my experience, it is often wise to use more than n decimal digits of precision when summing n terms of a series, computing nth-degree regressions, computing the zeros of an nth degree polynomial, etc."
> —Stoutemyer's Rule of Thumb

This kind of surprise is explored very thoroughly in [13], where this particular one is called "the tornado". The moral that we draw is twofold: first, that divergence doesn't always matter computationally (as we saw in the polylog example), and second that convergence doesn't always matter computationally either.

3 Recommendations

These remarks on the inutility of the dichotomy between convergence and divergence as $N \to \infty$, together with our observations that the students don't understand the concept anyway (though they can play the game of deciding whether this series is 'nardac' or whether it's 'clumglum' instead, by turning the crank on the ratio test, for example [9]), lead me to the most outrageous proposal in this paper: *Infinite series should be pushed out of the calculus curriculum.* Where, exactly, they should be pushed to is another matter.

Infinite series are good for demonstrating the existence of solutions; the notion of convergence has some applications in other areas, albeit rather fewer than a classically-trained mathematician might think. But in my opinion these benefits are not enough to warrant spending so much time in our courses on, especially in comparison with more obviously worthwhile topics, such as basic numerical analysis.

Truncated series or polynomials, on the other hand, have a definite place: polynomial approximation plays a huge role. This brings up my second opinion (doubtless less controversial): *More about polynomials should be taught.* An excellent source of material at the senior high-school or entering university level is [3]. I recommend, in addition to material more commonly taught now (such as the fundamental theorem of algebra), that the cubic and quartic formulae, the nonexistence of radical formulas for general polynomials of degree 5 or higher, polynomial interpolation, resultants and other connections to matrices, perhaps other polynomial bases (such as the Lagrange basis, the Chebyshev basis, and others), and much more all be taught.

I make this recommendation, even though polynomial rootfinding is done nowadays by the more useful and fundamental eigenvalues! The idea is that polynomials are accessible, and a natural route to eigenvalues; the reversal isn't so painful, really. Polynomials in and of themselves have tremendous applications (particularly in computer-aided geometric design), and burgeoning theory in the multivariate case, since the invention in the 1960's by Buchberger of a method to convert a given polynomial problem to one more easily dealt with, i.e. a Gröbner basis. I note that in the past decade it has been realized that eigenvalues are a vital tool for solving multivariate systems, too—analysis finds commuting families of matrices that have a common eigenstructure, and this is a good (possibly the best) way to solve the systems, once expressed in a Gröbner basis.

Probably my most important recommendation is to *include numerical analysis in the curriculum.* It isn't hard, when you stick to the two principles:

I. A good numerical solution gives you the exact solution to a nearby problem.

II. Some problems are sensitive to changes.

Checking how near the problem solved was is easy. For example, you plug in the computed root (or computed solution of the linear system) and see what's left over. This brings me to my fourth recommendation: *teach the students to check their results.* This is a part of numerical analysis, of course, but it is a keystone of using technology properly: we wish to avoid "Garbage in, Gospel out". Example areas: numerical integration, numerical solution of linear equations, numerical solution of nonlinear equations, the FFT, the singular value decomposition. See [5, 6] for detailed examples.

The second principle of numerical analysis is a marvelous excuse to work on derivatives. It has the added benefit of being necessary for scientists and engineers anyway: they have to study how sensitive their models are to errors in the input data.

I have other recommendations: minimize the optimization application of derivatives (sure, teach the students that the derivative is zero or nonexistent at an optimum, but let them graph it to see whether it's a max, min or saddle); lower the number of integration techniques taught (but keep *memory* and *partial fractions*; the first for efficiency but also mental health, and the second for applications such as in control theory). This allows concentration on integral formulation (i.e. modelling, aka reductionist thinking), surely a better way to spend one's time in possibly the only mathematics course one ever takes.

> "No proof without doubt." —E. J. Barbeau

> "I have absolutely no interest in proving things I know are true."
> —H. J. Abarbanel

My final recommendation is to *replace proof with programming.* Programming encourages and strengthens the same kind of thinking as does proof: precisionist grade, covering all the bases, breaking the problem up into small bits and putting them together. See the opinions of Doron Zeilberger on his home page (he points out that programming is just as much fun as proving, and even more so). As an extra benefit, classic principles such as induction become valuable tools for proving program correctness, with loop invariants. These applications have much greater immediacy for students than proving mathematical facts. They also have the advantage of being harder... and, when you relent and ask them to prove something simple about a mathematical formula by induction, they may view the exercise as being valuable!

My best example of shifting out proof is the way I teach the fundamental notion of continuity, and the equivalently fundamental notion of limits. I teach limits by means of continuity: I *declare* ("I tell you three times, and what I tell you three times is true.") that certain functions are continuous (polynomials, the exponential function, the sine and cosine function, and the logarithm away from $x < 0$ (and even there if limits are taken only horizontally)). Then they may use the definition of continuity $\lim_{x\to a} f(x) = f(\lim_{x\to a} x) = f(a)$ to "play the game" of evaluating limits by using the rules for sums, products, and division (which is the first one where they have to check). No proofs are involved: they simply play the game. *This is the way the game is played in all calculus courses, anyway, no matter what the instructor may think.* I just make the rules explicit.

I don't waste their time (or mine) trying to prove the rules, or to prove that the basic functions are continuous (after all, I don't even give them the ε–δ definition). I do use computation to evaluate some limits numerically: $\lim_{x\to 0}(e^x - 1)/x$, $\lim_{x\to 0} \sin(x)/x$, and $\lim_{n\to\infty}(1 + x/n)^n$ for numerical x from which they are expected to induce that the limit is e^x (this requires careful timing: if they see an example with n so large that $1 + x/n$ rounds to 1, then the computation gives 1 as the answer—and displacing that incorrect but oh-so-plausible answer is a job and a half, let me tell you).

Don't get the impression that I am against rigour: far from it. For example, when I teach complex numbers, for example, I absolutely do not start with the false-to-fact "Let i be the square root of -1". They know very well that there is no such number. This type of false supposition can lead to total confusion, or a bankrupt pragmatism on the part of the student. The same type of reasoning leads to contradictions, such as Perron's paradox: "Let N be the largest integer. Then if $N > 1$ we would have $N \cdot N > N \cdot 1$ or $N^2 > N$, a contradiction. Therefore $N = 1$."

Instead I define (as Gauss did) multiplication of number pairs, show that $(0, 1)^2 = (-1, 0)$, and therefore define $i = (0, 1)$. This is the way the HP calculator does it (as does Maple internally). Rigorous, but concrete: there is no argument, and the process is totally convincing. Students should learn rigour by example as well as by precept.

Students are expected to use complex numbers in a seamless fashion; they are expected to "play the game" of evaluating limits by using continuity; and they are expected to write and understand small programs for the evaluation of finite sums such as are used in quadrature: left hand Riemann sums, right-hand Riemann sums, the midpoint rule, and the trapezoidal rule. We analyze the error behaviour by showing that for monotonic functions, the left and right hand *finite* Riemann sums bound the true answer on either side; and for convex functions that the midpoint and trapezoidal rules bound the true answer on either side; and the students are very easily convinced in a concrete fashion by experimentation that they may get better (more accurate) answers by taking more subdivisions. This is the beginning point for infinite series, and for analysis: it is my opinion that no more than this should be seen in a first course. The technical and philosophical details, together with the hard inequalities and analysis, should be left for a later course.

4 Concluding remarks

We have taught calculus with technology since 1988. Our students are taught the following 'Three Laws' of equipment use:

1. Any tool should always be used to *expand the user's capabilities*, and not as a crutch to prop up weak skills.

2. One must be *smarter than one's equipment*, knowing its limitations thoroughly.

3. A piece of equipment is *not a substitute for thought*. The user will always be *responsible* for what is going on, even if the details are carried out 'under the hood'.

These laws do not depend on particular hardware or software: they are rules for computer-mediated thinking.

Finally, calculus has *unintentional power* [15]. Changing this basic piece of curriculum affects all subsequent courses, and a significant portion of the educated populace, and it's hard to predict just how the effects will show up. I would like to think of the recommendations in this paper as a starting point for discussion. Something has to give, if we add more material to the curriculum; something has to go. My vote is for series, though I am sure I have not anticipated all effects. I have thought about the fact that this is the first place students see infinity[2], and that (at some point) they have to argue about whether $0.99999\ldots$ is equal to 1 or not. Series is also the first place that they manipulate integer-valued functions (the formulae for the nth terms). It is the second place that they work with inequalities (the first is in the classical treatment of $\lim_{x\to 0}\sin(x)/x$, which I have also removed). Removing series from the curriculum will have side-effects, that they have less practice with infinity and with inequalities. I am prepared to accept them, because I believe that the material I have replaced it with has as many benefits, and some greater.

One of the most useful (and invisible) effects of calculus has been to encourage *reductionist* thinking (one of the most successful ideas ever). This happens when people formulate integrals (in sections typically called "applications of integration", which can degenerate into formula memorization but in skilled hands does not). Calculus also encourages linear precisionist-grade thinking, clear-edged logic, clarity and economy of hypotheses, and *constructivist* thinking, which the approach of this paper encourages further. Calculus also gives students a chance (and a reason) to acquire *algebraic fluency*, without which they are unable to succeed in subsequent courses. "I never used that stuff again"—a common quote, but dead wrong. Calculus affects how people think.

Now it has changed.

Acknowledgements

Many people, good teachers and good students all, have helped me to form my opinions over the years. I owe particular gratitude to David Jeffrey, Paul Sullivan, Martin Müser and my other colleagues in the Applied Mathematics Department at Western for many stimulating conversations. Sorin Costreie and Jim Morey, Ph.D. students in Philosophy and in Computer Science respectively at Western, read an early draft of the paper and made valuable comments. Samantha Brennan (currently Chair of the Department of Philosophy at Western) pointed out the work of Peter Danielson to me. Contact with designers of computer algebra systems, such as Keith Geddes of Maple and David Stoutemyer and Al Rich of Derive, has helped me to develop my own skills at 'computer-mediated thinking'. My mistakes are of course my own. I am also grateful to Michel Beaudin, Gilles Picard, and Kathleen Pineau for the invitation to express these opinions in this forum.

[2] This is an important pedagogical point, and not to be taken lightly

References

[1] K. Appel and W. Haken. Every planar map is four colorable. part I. discharging. *Illinois J. Math.*, 21:429–490, 1977.

[2] K. Appel and W. Haken. Every planar map is four colorable. part II. reducibility. *Illinois J. Math.*, 21:491–567, 1977.

[3] E. J. Barbeau. *Polynomials*. Springer-Verlag, 1989.

[4] Jonathan M. Borwein and Peter B. Borwein. Strange series and high precision fraud. *American Mathematical Monthly*, 99(7):622–640, Aug.-Sep. 1992.

[5] Robert M. Corless. Six, lies, and calculators. *The American Mathematical Monthly*, 100(4):344–350, April 1993.

[6] Robert M. Corless. Variations on a theme of Newton. *Mathematics Magazine*, 71(1):34–41, Feb 1998.

[7] Robert M. Corless. Generalized companion matrices in the Lagrange basis. In Laureano Gonzalez-Vega and Tomas Recio, editors, *Proceedings EACA*, pages 317–322, June 2004.

[8] Robert M. Corless, Gaston H. Gonnet, D. E. G. Hare, David J. Jeffrey, and Donald E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5:329–359, 1996.

[9] Robert M. Corless and David J. Jeffrey. Scientific computing in mathematics education: One part of the revolution. *J. Symb. Comp.*, 23:485–495, 1997.

[10] Robert M. Corless and Stephen M. Watt. Bernstein bases are optimal, but, sometimes, Lagrange bases are better. Technical Report TR-0x-04, ORCCA, www.orcca.on.ca/Techreports, July 2004.

[11] Andrea di Sessa. *Changing Minds: Computers, Learning and Literacy*. Bradford Books, October 2001.

[12] Barbara S. Edwards and Michael B. Ward. Surprises from mathematics education research: Student (mis)use of mathematical definitions. *American Mathematical Monthly*, 111(5):411–425, May 2004.

[13] G. Christopher Essex, Matt Davison, and Christian Schultzky. Numerical monsters. SIGSAM BULLETIN*: Communications in Computer Algebra*, 34(4):16–32, 2000.

[14] Jo Handelsman, Diane Ebert-May, Robert Beichner, Peter Bruns, Amy Chang, Robert DeHaan, Jim Gentile, Sarah Lauffer, James Stewart, Shirly M. Tilghman, and William B. Wood. Scientific teaching. *Science*, 304:521–522, April 23 2004.

[15] Chuck Huff. Unintentional power in the design of computer systems. *SIGCAS Comput. Soc.*, 26(4):6–9, 1996.

[16] Peter Jones. CAS and the teaching and learning of mathematics: towards the intelligent partnership. In *this Proceedings*, 2004.

[17] William Kahan. Mathematics written in sand. In *Proceedings of the American Statistical Association*, pages 12–26, 1983.

[18] Uri Leron. Mathematical thinking & human nature: Consonance & conflict. Technical report, Technion, 2004.

[19] L. Lewin. *Polylogarithms and Associated Functions.* North Holland, Amsterdam, 1981.

[20] O.A.R. Mahroo and T. D. Lamb. Recovery of the human photopic electroretinogram after bleaching exposures: estimation of pigment regeneration kinetics. *J. Physiology*, 554(2):417–437, January 2004.

[21] Pirooz Mohazzabi and Thomas A Fournelle. Evaluation of ill-behaved power series. *American Mathematical Monthly*, 111(4):308–321, April 2004.